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Phase Diagram of the One-State Potts Model on the Cayley Tree

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The phase diagram of the one-state Potts model on the closed asymmetric Cayley tree with branching ratio $r=2$ is obtained from the Bethe–Peierls map. The route to chaos, via the period doubling cascade, is obtained by considering the antiferromagnetic coupling limit. The connection of the Potts model with the percolation problem is shown by calculating the order parameter, its susceptibility, the internal energy, and the specific heat as well as their asymptotic behavior at the paramagnetic–ferromagnetic critical point. Due to the type of the lattice and to the polynomial character of the map, this is the simplest known example of a McKay–Berker–Kirkpatrick spin-glass.

KEY WORDS: Bethe–Peierls map; Möbius transformation; logistic map; mean-field percolation; McKay–Berker–Kirkpatrick spin-glass.

In this paper we study the one-state ($p=1$) Potts model on the closed asymmetric Cayley tree. It is shown that for a general p the renormalization group transformation (Bethe–Peierls map) is a rational recursion relationship relating the effective magnetic field in two consecutive generations. For the one-state model the Bethe–Peierls map reduces to a polynomial whose degree is equal to the branching ratio r. For even r the map is unimodal, displaying a route to chaos via bifurcation,[(1)] while for odd r, as well as in the limit $r \to \infty$, one has a homeomorphism which can have no periodic point with period greater than two.[(2)] The particular case of $r=2$, the only one which can be treated analytically, is studied in detail.

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The study of exactly soluble frustrated spin systems defined on hierarchical lattices as prototypes of spin-glasses was initiated in 1982 by McKay *et al.* (MBK)[3] and extended further by Derrida *et al.*[4] who presented a detailed study of the analytic behavior of the free energy. MBK analyzed an Ising model with nonrandom competitive interactions whose one-dimensional recursive relation for the renormalized bond strength has five parameters, all of them related with the geometric features of a quite uncommon and artificial lattice. Fixing four chosen parameters, they discussed the renormalization group orbits as a function of the remaining one, which measures the frustration present in the system. As this parameter is varied, increasing the frustration, the usual period doubling route to chaos appears, and they interpreted the chaotic regime as a spin-glass phase of the system.

The presence of chaos in another Ising system also with nonrandom competitive interactions but defined on the familiar Cayley tree has been investigated by Yokoi *et al.*[5] However, their model generates a two-dimensional mapping. In contrast to these models, our system has no competitive interactions and its one-dimensional map is parametrized by the temperature T, by the external field H_1, by the branching ratio r, and by the number of states p, which measures the amount of frustration present. Therefore, considering the familiarity with the lattice, and the nature of the parameters, in particular, the special role played by p as the frustration knob, our system forms a new type and it is the simplest example of an MBK spin-glass.

The nearest-neighbor p-state Potts model on the Cayley tree is known to have a trivial (open chain) partition function unless an external magnetic field is turned on.[6,7] In the presence of a weak magnetic field the model exhibits, as a consequence of the surface effects, a phase transition of continuous order. This transition is characterized by a field term in the free energy whose exponent varies continuously with the temperature.[8,9] On the other hand, subtracting the surface effects, we can study the local properties deep inside the tree which reproduce the results of the mean field (Bethe–Peierls) approximation.[10] In this case the model displays a phase transition of first or second order, depending on p.[11] A comprehensive treatment of both global and local thermodynamic properties can be obtained by the introduction of an additional (ghost) spin[12–15] which interacts with all spins in the interior and on the surface of the Cayley tree. In this paper we shall deal only with mean field (local) results. We notice that even though it is now over 30 years since the Potts model was first proposed, its mean field approximation has not yet been fully developed to the same extent as that of the Ising model. To the best of our knowledge, this is the first study of the one-state model on the Cayley tree. This system has a very rich thermodynamic behavior and the complete understanding of its features may increase our knowledge of polynomial maps. For example, from the fact that the critical exponents are independent of the polynomial degree one can infer the existence of some universal geometric features in the Mandelbrot and Julia sets. Further interest in this system stems from the known result that the statistics of a percolating cluster on a lattice is described by the one-state Potts model.[16] The procedure followed is to obtain first the results for a general p and then to take the limit $p \to 1$. As shown later, in this limit one recovers directly the mean field results of the percolation problem.

Using the Mittag–Stephen[17] representation for the Potts variable, we write the Hamiltonian defining the system as

$$H = -pJ \sum_{\langle ij \rangle} \sum_{q=0}^{p-1} \lambda_i^q \lambda_j^{p-q} - pH_1 \sum_i \sum_{q=0}^{p-1} \lambda_i^q \lambda_g^{p-q} \tag{1}$$

where the summation is over all pairs $\langle i, j \rangle$ of nearest neighbor sites, the second sum is over all sites i in the tree, and λ_g is the ghost spin. Freezing this spin in any one of its p-states, one recovers the Hamiltonian for the system on the open tree in the presence of the external magnetic field H_1. The edges representing the interactions of the ghost with the other spins change drastically the most relevant topological features of the open loopless lattice by transforming it into the closed asymmetric Cayley tree (c.a.t.); see Fig. 1.

The correlation function $\langle \lambda_i^q \lambda_j^{p-q} \rangle$ is given by

$$\langle \lambda_i^q \lambda_j^{p-q} \rangle = (1 - e^{-p\beta J_{ij}})/[1 + (p-1)\, e^{-p\beta J_{ij}}] \tag{2}$$

where $\langle \cdot \rangle$ stands for the usual Gibbs average. The rhs of Eq. (2) is the thermal transmissivity[18] $t(J_{ij})$ between the (external) spins at sites i and j. The thermal transmissivity $t(J)$ is the most natural variable because of its simple composition rule. For instance, if three spin variables at sities i, j, and k are connected in series by the coupling constants J_{ij} and J_{jk}, the equivalent transmissivity $t(J_{eq})$ between the external spins is the product $t(J_{ij})\, t(J_{jk})$. On the other hand, if two spins at sites i and j are coupled in parallel by the coupling J_{ij} and K_{ij}, the equivalent (dual) transmissivity $t^d(J_{eq})$ is the product $t^d(J_{ij})\, t^d(K_{ij})$, where

$$t^d(J_{ij}) = \exp\ -p\beta J_{ij} \tag{3}$$

Applying now the composition rules to the transmissivities in the elementary generating cluster (Fig. 1) of the tree, one obtains the equiva-

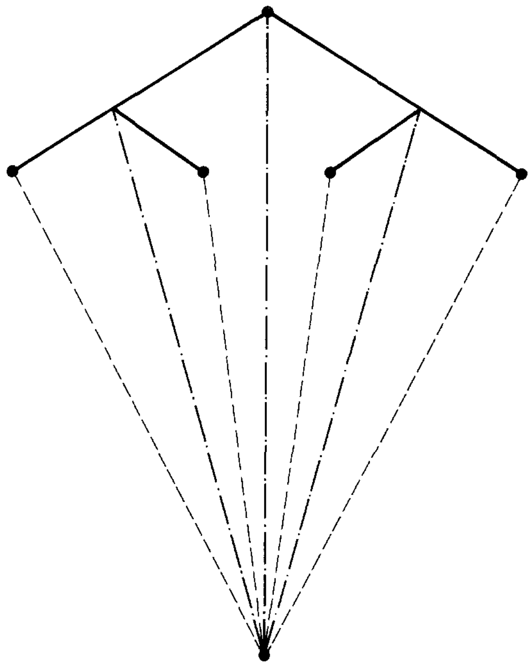

Fig. 1. A closed asymmetric Cayley branch ratio $r = 2$ with two generations. The branch with one more generation is obtained by replacing each dashed edge by the elementary cluster (cut diamond) which is formed by two solid edges representing the coupling J, two dashed edges representing the renormalized magnetic field H_N, and by the dot-dashed edge (bare magnetic field) connecting the ghost spin with the top spin in the diamond. The BP map given by Eq. (4) is obtained by calculating the effective interaction between the top and ghost spins in the cut diamond. The c.a.t. is formed by connecting the top sites of three branches to an extra (central) site.

lent transmissivity X_{n+1} between the ghost and any spin in the $n+1$ shell as a recursion relationship, the so-called Bethe–Peierls (BP) map:

$$X_{n+1} = B(X_n : t, h_1, 2)$$
$$= \frac{h_1 + 2[1 + (p-2) h_1] tX_n + [(p-2) + (p^2 - 3p + 3) h_1] t^2 X_n^2}{1 + 2(p-1) h_1 tX_n + (p-1)[1 + (p-2) h_1] t^2 X_n^2} \tag{4}$$

The transmissivities h_1 and X_n are obtained from Eq. (2), substituting, respectively, J_{ij} by the external (bare) magnetic field H_1 and by the renormalized field H_n acting upon the spins in the nth shell.

In the limit $p \to 1$ the BP map reduces to a polynomial of degree two which can be conjugated to the Mandelbrot map $M(X) = X^2 + C$ via a Möbius transformation,[19]

$$M(X) = L \circ B \circ L^{-1}(X), \qquad L(X) = aX + b \tag{5}$$

where

$$a = -(1-h_1)\,t^2, \qquad b = (1-h_1)\,t \tag{6}$$

$$C = t(1-t)(1-h_1) \tag{7}$$

Equation (7) allow us to draw the phase diagram, since the knowledge of the critical values C_{2^s} of the parameter where the orbits of the M map become indifferent gives the equations of the transition lines. Let us consider first the values of C outside (the real part of) the Mandelbrot set, i.e., $C \notin (-2, 1/4)$. In this case all points in the X plane, except those on the Julia set, are attracted to infinity, which is the only (super) stable attractor of the map, yielding a thermodynamically ill-defined system. In the h_1–t parameter space this will correspond to the regions $I_{1/4}$, I_{-2} (Fig. 2). In the

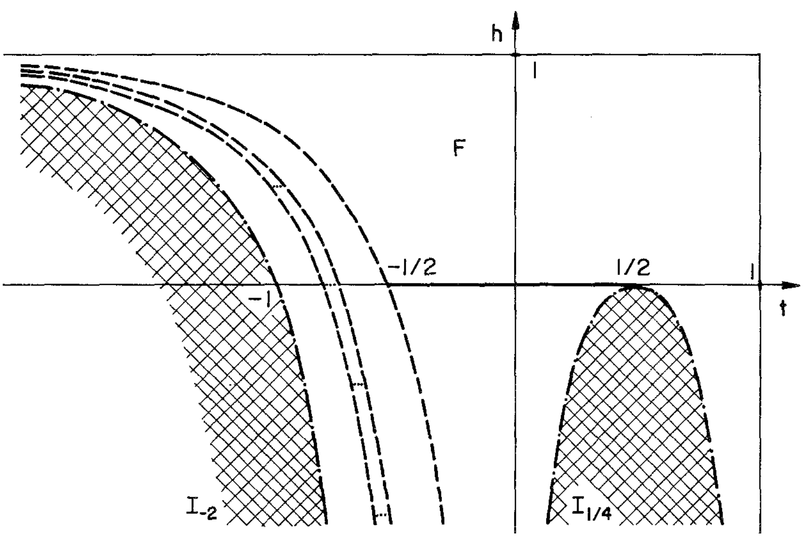

Fig. 2. In the regions $I_{1/4}$ and I_{-2} the system is thermodynamically ill defined. Their boundaries (dot-dashed lines) are obtained by setting $C = 1/4$, $C = -2$ in Eq. (7). In the region denoted by F the stable fixed point is X^*_+. The paramagnetic fixed point $X^*_- = 0$ is stable in the region $h_1 = 0$, $t \in (-1/2, 1/2)$. In the first band (from right to left) the system is in a period-2 cycle. The intermediary strip represents the succession of period-2^s cycle bands generated by the period doubling cascade. The last band at the right end is the (chaotic) spin-glass band.

interval $-3/4 < C < 1/4$ the M map has a stable fixed point which is associated by the Möbius transformation to the fixed points

$$X_{\pm}^{*} = \frac{1}{2t^2(1-h_1)} \{2t(1-h_1) - 1 \pm [1 + 4t(t-1)(1-h_1)]^{1/2}\} \tag{8}$$

of the BP map. In the h_1–t space, X_{+}^{*} is stable in the region denoted by F, which excludes the straight line segment $h_1 = 0$, $t \in (-1/2, 1/2)$, where the paramagnetic fixed point $X_{-}^{*} = 0$ is the stable one. Looking back to the tree, this corresponds to the case when the correlation function between the ghost and the spins infinitely far from the surface does not change when one moves inward the interior of the tree. On the other hand, for $-5/4 < C < -3/4$ the M map has a stable period-2 cycle.

This will give rise to the first band (from right to left) in the h_1–t plane. Its boundaries are obtained from Eq. (7) by setting C equal to $-3/4$ and $-5/4$, respectively. In the tree the correlation between the ghost and the spins belonging to two consecutive generations will alternate, taking the values given by the cycle. Lowering further, but keeping the value of C inside the Mandelbrot set, one obtains the usual period doubling route to chaos. Taking C in the interval $(C_{2^s}, C_{2^{s+1}})$, where the period-2^s cycle is stable, will give rise to the period-2^s band in the BP parameter space. The onset of the chaotic behavior of the M map occurs for $C = C_{\text{ch}} \cong -1.42...$ and according to the conjugation given above the chaotic band boundaries are obtained by setting $C = C_{\text{ch}}$ and $C = -2$. Except for the periodic windows inside the chaotic band, the correlation functions between the ghost and the spins deep inside the tree are random. The microscopic interpretation given by MBK to the chaotic regime can be easily extended by noting that the subset of noncontiguous spins which are strongly correlated with each other now form rings in the tree like those in a sequoia tree.

Let us show how the correspondence between the bond percolation problem and the one-state Potts model pointed out by Kasteleyn and Fortuin can be verified explicitly in this case. Let us construct first the thermal transmissivity m between the ghost and the central spin of the Cayley tree in the region of the h_1–t plane where $X_{\pm}^{*}$ are stable:

$$m = X_{n+1} + tX_n(1 - X_{n+1}) \tag{9}$$

In the (thermodynamic) limit of $n \to \infty$ we can replace the X_n' by the fixed points $X_{\pm}^{*}$ so that at zero field the order parameter expression reduces to

$$m = \begin{cases} 0, & |t| < 1/2 \\ 1 - [(1-t)/t]^3, & 1/2 < t < 1 \end{cases} \tag{10}$$

We observe now that if we equate t with the probability π of a bond being present, therefore restricting our results to the region of $t \geqslant 0$, the above expression for m is identical to the ratio of the percolation probability $P(\pi)$ by π,[20] i.e.,

$$m(t, 0) \equiv P(\pi)/\pi \tag{11}$$

The order parameter susceptibility at $h_1 = 0$ is given by

$$\beta^{-1}\chi = \begin{cases} (1+t)/(1-2t), & t < 1/2 \\ (1-t)^2\,[t(1+t) - 2(2t-1)]/[t^3(2t-1)], & t > 1/2 \end{cases} \tag{12}$$

We remark that the high-temperature ($t < 1/2$) expression for $\beta^{-1}\chi$ coincides with the ratio of the mean finite cluster size $S(\pi)$ by π.[20] From Eqs. (9), (10), and (12) we get that $\beta = 1$, $\gamma = \gamma' = 1$, $\delta = 2$, which are the mean field critical exponents of the bond percolation problem.[21] We now turn our attention to the internal energy (per number of pairs), which is obtained by calculating the thermal transmissivity between the central spin and a top spin[22] and zero-field specific heat,

$$U = -J[t + (1-t)\, X_n X_{n+1}] \tag{13}$$

$$C/k_B = \begin{cases} (\beta J)^2\,(1-t), & t < 1/2 \\ (\beta J)^2\,(1-t)[(2t-1)(2t^2 - 7t + 4) + t^5]/t^5, & t > 1/2 \end{cases} \tag{14}$$

The expansion around $t_c = 1/2$ yields a linear cusp behavior for C/k_B, so that $\alpha = \alpha' = -1$, which are the mean-field percolation critical exponents.[21]

In conclusion, we have presented the mean field phase diagram of the $p = 1$ Potts model on the c.a.t. A Möbius transformation allow us to conjugate the BP map with the quadratic map, yielding a relationship between the t and h_1 which determines the phase diagram of the model. Stressing the fact that the main result is the Bethe–Peierls map and all other results follow directly from it, we show that the model has a sound physical content by establishing its connections with the percolation problem. This is accomplished by calculating the $p = 1$ thermodynamic properties in the phases determined by $X_{\pm}^*$ and in the critical region. The identification of $t(J)$ with the bond probability is sufficient to recover the percolation functions and its critical exponents. Therefore, the procedure used automatically satisfies the requirements of the Kasteleyn–Fortuin theorem. We also observe that the percolation results can be analytically continued to the whole region right of the critical line $-3/4 = t(1-t)(1-h_1)$.

REFERENCES

1. P. Collet and J. P. Eckmann, *Iterated Maps on the Interval as Dynamical Systems* (Birkhauser, Cambridge, Massachusetts, 1980).
2. R. L. Devaney, *An Introduction to Chaotic Dynamical Systems* (Benjamin Cummings, Boston, Massachusetts, 1986.
3. S. R. McKay, A. N. Berker, and S. Kirkpatrick, *Phys. Rev. Lett.* **48**:767 (1982).
4. B. Derrida, J. P. Eckmann, and A. Erzan, *J. Phys. A* **16**:893 (1983).
5. C. S. O. Yokoi, M. J. de Oliveira, and S. R. Salinas, *Phys. Rev. Lett.* **54**:163 (1985).
6. T. P. Eggarter, *Phys. Rev. B* **9**:2989 (1974).
7. H. G. Baumgärtel and E. Müller-Hartmann, *Z. Phys. B* **46**:227 (1982).
8. E. Müller-Hartmann and J. Zittartz, *Phys. Rev. Lett.* **33**:893 (1974).
9. Y. K. Wang and F. Y. Wu, *J. Phys. A* **9**:593 (1976).
10. T. Kihiara, Y. Midzuno, and J. Shizume, *J. Phys. Soc. Jpn.* **9**:681 (1954).
11. F. Y. Wu, *Rev. Mod. Phys.* **54**:235 (1982).
12. P. L. Christiano and S. Goulart Rosa, Jr., *Phys. Lett. A* **101**:275 (1984); *J. Phys. C* **18**:L407 (1985); *Phys. Rev. A* **34**:730 (1986).
13. F. A. Bosco and S. Goulart Rosa, Jr., *Europhys. Lett.* **4**:1103 (1987).
14. F. S. de Aguiar, F. A. Bosco, and S. Goulart Rosa, Jr., *Phys. Lett. A* **127**:194 (1988).
15. A. S. Martinez and S. Goulart Rosa, Jr., *Phys. Rev. A* **38**:4304 (1988).
16. P. W. Kasteleyn and C. M. Fortuin, *J. Phys. Soc. Jpn.* (Suppl.) **26**:11 (1969); *Physica* **57**:536 (1972).
17. L. Mittag and M. J. Stephen, *J. Math. Phys.* **12**:441 (1971).
18. C. Tsallis and S. V. F. Levy, *Phys. Rev. Lett.* **47**:950 (1981).
19. P. Blanchard, *Bull. Am. Math. Soc.* **11**:85 (1984).
20. D. Stauffer, *Introduction to Percolation Theory* (Taylor and Francis, London, 1985.
21. J. W. Essam, *Rep. Prog. Phys.* **43**:834 (1980).
22. P. L. Christiano, Ph.D. thesis, IFQSC, Universidade de São Paulo (1985), unpublished.

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